The goal of the following section is to estimate $\mu$, population mean, using a sample mean $\bar{X}$ when the sample size $n < 30$ and population standard deviation $\sigma$ is unknown. This method is also used for any size of sample when population standard deviation $\sigma$ is unknown and cannot be approximated. This method works only in the case when the measured (counted) phenomenon is at least approximately normally distributed (bell shaped symmetrical mound of relative frequencies).

The probability theory says that in the case above

$$\frac{\bar{X} - \mu}{s \sqrt{n}},$$

the so called $t$ – value, is distributed as *Student’s t distribution with $n – 1$ degrees of freedom*.

The tables for this distribution are well known. The graph for the density is very similar to the one for normal distribution but with somewhat thicker tails.
Here is a comparison with standard normal density curve (dotted blue), using Student’s $t$ distribution with 1, 3, and 11 degrees of freedom (the higher degrees the taller the density curve):

**Student’s $t$ density versus Z density**

**Usage**

Once again we want to find an interval about $\bar{X}$ so that we can say, with $1-\alpha$ level of confidence (probability, chance), that the population mean $\mu$ is in that interval. Corresponding critical $t$ – values $t_{\alpha/2}$ now depend on the degrees of freedom $(n - 1)$. 
The (1-\(\alpha\))-confidence interval for the population mean \(\mu\) is (so called \(t\)-interval)

\[
P \left[-t_{\alpha/2} < \frac{\bar{X} - \mu}{s/\sqrt{n}} < t_{\alpha/2}\right] = 1 - \alpha
\]

\[
\Rightarrow P \left[\bar{X} - t_{\alpha/2} \frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2} \frac{s}{\sqrt{n}}\right] = 1 - \alpha
\]

This (1-\(\alpha\))-confidence interval for the population mean \(\mu\) is (so called \(t\)-interval)

\[
\bar{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}
\]

And once again, maximum error of estimate

\[
MoE = t_{\alpha/2} \frac{s}{\sqrt{n}}
\]

is also called the margin of error respective to the confidence level 1–\(\alpha\).

***

Example 1.

For a certain type of prehistoric lizard there are 10 skeletons available. The average length of the skeletons is 11.6 ft while the standard deviation of the sample is 2.8 ft. What would the 95% confidence interval for the mean length of the lizard?

Solution:

We assume normal distribution for the length thus using \(t\)-interval formula above with 9 degrees of freedom, \(t_{0.025} = 2.262\).
\[ \bar{X} \pm t_{a/2} \frac{s}{\sqrt{n}} = 11.6 \pm 2.262 \frac{2.8}{\sqrt{10}} = 11.6 \pm 2 \]

We can say that we are about 95% sure the lizard was somewhere between 9.6 and 13.6 long, on average. Margin of error is approximately 2.

Example 2.

Andre is head waiter at a famous restaurant. IRS is doing an audit of his tax return and he was required to submit a sample of 8 credit card bills indicating his tip. The results were

\$10.00 \$12.75 \$11.93 \$11.15 \$15.70 \$14.50 \$9.10 \$13.65

Find the 90% confidence interval for the population mean of Andre’s tips assuming normal distribution for the tips. What is the margin of error?

Solution:

We have \( \bar{x} = 12.3475 \) and \( s \approx 2.2487 \) (calculate for homework). There are 7 degrees of freedom and the corresponding critical \( t \)– value is \( t_{0.05} = 1.895 \). The margin of error is

\[ MoE = t_{a/2} \frac{s}{\sqrt{n}} = 1.895 \cdot \frac{2.2487}{\sqrt{8}} \approx 1.5066 \]

Therefore the 90% interval of confidence is

\[ (\bar{x} - MoE, \bar{x} + MoE) = (10.8409, 13.8541) \].
Example 3.

The book “Good Cholesterol, Bad Cholesterol,” by Roth and Streicher, gives the data from 8 popular fast-food restaurants about the amount of calories in 3 ounces of french fries.

\[ 222 \quad 255 \quad 254 \quad 230 \quad 249 \quad 222 \quad 237 \quad 287 \]

Use the data to find the 99% confidence interval for the mean calorie count in 3 ounces of french fries obtained from fast-food restaurants. Assume the normal distribution for the calorie count in 3 ounces of French fries.

Solution:

First we need to calculate sample mean and sample standard deviation. This is \( \bar{x} = 244.5, \ s = 21.732 \).

\[
244.5 \pm 3.499 \frac{21.732}{\sqrt{8}} = 244.5 \pm 26.884 .
\]

Approximately (217.6, 271.4). The maximum error of estimate is 26.884.

**Homework:** Check online.