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MA07: Algebra-I

Final Exam: Solutions and Answers

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Problem 1. Two points on the plane are $(-2,\ 5)$ and $(3,\ -1)$.

(1). Find the slope of the line that joins the above two points.

Space for your solution:

We know two points on the line in question. Let's *start* at one of those points and go until we *finish* at another point. As we go, both the x and the y coordinates of our moving point will change. By definition, the slope of the line is the fraction

$$\textbf{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{(y \text{ of finish}) - (y \text{ of start})}{(x \text{ of finish}) - (x \text{ of start})}.$$

(The slope will not depend on the choice: which one of the two points is the start, and which one — the finish.) In our case, let's choose the point $(-2,\ 5)$ as the start and $(3,\ -1)$ — as the finish. Then

$$\textbf{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{(y \text{ of finish}) - (y \text{ of start})}{(x \text{ of finish}) - (x \text{ of start})} = \frac{(-1) - (5)}{(3) - (-2)} = \frac{-6}{5} = -\frac{6}{5}.$$

(2). Find the slope-intercept equation of the line that joins the above two points.

Space for your solution:

A point $(x,\ y)$ is on the line that joins the two points $(-2,\ 5)$ and $(3,\ -1)$ if and only if we get the same slope when going from $(-2,\ 5)$ to $(3,\ -1)$ as when going from $(-2,\ 5)$ to $(x,\ y)$:

$$\frac{(-1) - (5)}{(3) - (-2)} = \frac{(y) - (5)}{(x) - (-2)}$$

Simplify the above equation by computing the arithmetic expressions:

$$\frac{-6}{5} = \frac{y - 5}{x + 2}.$$

Cross multiply: $(-6)(x+2) = (y-5)\cdot 5 \quad \Leftrightarrow$ (open parentheses) $-6x - 12 = 5y - 25 \quad \Leftrightarrow$ (combine the like terms by adding 25 to both sides) $-6x + 13 = 5y \quad \Leftrightarrow$ (express y in terms of everything else by dividing both sides by 5)

$$y = -\frac{6}{5}x + \frac{13}{5}.$$

(The slope of the line is $-\frac{6}{5}$ and the y-intercept is $\frac{13}{5}$.)

Problem 2. Divide the polynomial $x^4 - 7x^2 + x$ by the polynomial $x^2 - 2x + 1$.

Space for your solution:

(Don't forget to insert the missing degrees of x before doing long division.)

$$
\begin{array}{r|l}
 & x^2 + 2x - 4 \\
\hline
x^2 - 2x + 1 & x^4 + 0x^3 - 7x^2 + x + 0 \\
 & x^4 - 2x^3 + x \\
\hline
 & 2x^3 - 8x^2 + x \\
 & 2x^3 - 4x^2 + 2x \\
\hline
 & -4x^2 - x + 0 \\
 & -4x^2 + 8x - 4 \\
\hline
 & -9x + 4
\end{array}
$$

Problem 3. Using the result of the preceding problem, write $\frac{x^4-7x^2+x}{x^2-2x+1}$ as a sum of a polynomial and a "proper fraction", i.e. a fraction of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials and the degree of f(x) is strictly less than the degree of $g(x)$.

Space for your solution:

The above long division means that

$$\frac{x^4 - 7x^2 + x}{x^2 - 2x + 1} = x^2 + 2x - 4 + \frac{-9x + 4}{x^2 - 2x + 1}.$$

Problem 4. Solve the system of linear equations:

$$\begin{cases} 2x + y = 5 \\ x + y = 2 \end{cases} .$$

Space for your solution:

(This solution follows the substitution method.)

$\begin{cases} 2x + y = 5 \\ x + y = 2 \end{cases}$ $\Leftrightarrow$ (using second equation, express y in terms of x and constants; then substitute that expression instead of y in the first equation) $\begin{cases} y = 2 - x \\ 2x + (2 - x) = 5 \end{cases}$ $\Leftrightarrow$ (open parentheses and combine the like terms in the second equation) $\begin{cases} y = 2 - x \\ x + 2 = 5 \end{cases}$ $\Leftrightarrow$ (find the value of x from the second equation and then substitute that value instead of x in the first equation) $\begin{cases} y = 2 - 3 = -1 \\ x = 3 \end{cases}$.

Problem 5. Factor completely the polynomial $x^2y^3 - 6xy^2 + 9y$.

Space for your solution:

First of all factor out the **G**reatest **C**ommon **D**ivisor of the polynomials x^2y^3, $6xy^2$ and $9y$ — which is y:

$$x^2y^3 - 6xy^2 + 9y = y \cdot (x^2y^2 - 6xy + 9).$$

Next, let's try to fit the polynomial $x^2y^2 - 6xy + 9$ into the template of the binomial formula:

$$A^2 + 2AB + B^2 = (A + B)^2.$$

From $A^2 = x^2y^2$ we get $A = xy$. From $2AB = -6xy$ we get $AB = -3xy$. Since $A = xy$, we can cancel A on the left and xy on the right of the identity $AB = -3xy$ and get $B = -3$. The next step is to check that $B^2 = 9$. It works! Therefore we can use the binomial formula and write $x^2y^2 - 6xy + 9 = (xy - 3)^2$.

Combining all the factorizations together, we get:

$$x^2y^3 - 6xy^2 + 9y = y \cdot (x^2y^2 - 6xy + 9) = y \cdot (xy - 3)^2.$$

Problem 6. Solve the equation:

$$\frac{2}{x-3} + \frac{4}{x+3} = \frac{4x}{x^2-9}.$$

Space for your solution:

We have to find the **L**east **C**ommon **M**ultiple of the three polynomials $x-3$, $x+3$ and x^2-9. Since $x^2-9=(x-3)(x+3)$, the polynomial x^2-9 is the LCM. After we multiply the fractions above by their respective adjoint factors, we will get them all have the same denominator:

$$\frac{2}{x-3} + \frac{4}{x+3} = \frac{4x}{x^2-9} \quad \Leftrightarrow \quad \frac{2(x+3)}{(x-3)(x+3)} + \frac{4(x-3)}{(x+3)(x-3)} = \frac{4x}{x^2-9} \quad \Leftrightarrow \quad \frac{2(x+3)+4(x-3)-4x}{(x-3)(x+3)} = 0 \quad \Leftrightarrow$$
$$\frac{2x+6+4x-12-4x}{(x-3)(x+3)} = 0 \ \Leftrightarrow \ \frac{2x-6}{(x-3)(x+3)} = 0 \ \Leftrightarrow \left\{ \begin{array}{l} 2x-6=0 \\ (x-3)(x+3) \neq 0 \end{array} \right. \quad \Leftrightarrow \left\{ \begin{array}{l} 2x=6 \\ (x-3)(x+3) \neq 0 \end{array} \right.$$
$$\Leftrightarrow \left\{ \begin{array}{l} x=3 \\ (x-3)(x+3) \neq 0 \end{array} \right. \quad \Leftrightarrow \ x \in \emptyset.$$

The last conclusion (no solution) is made because the conditions $x=3$ and $(x-3)(x+3) \neq 0$ are contradictory. Indeed, if we substitute $x=3$ into $(x-3)(x+3)$ we do get zero.

Problem 7. Using properties of exponents, write the following fraction as product of exponents:

$$\frac{x^5y^3}{y^2x^{10}\sqrt{2z}}.$$

Space for your solution:

$$\frac{x^5y^3}{y^2x^{10}\sqrt{2z}} = x^{5-10}y^{3-2}(\sqrt{2z})^{0-1} = x^{-5}y^1\left((2z)^{\frac{1}{2}}\right)^{-1} = x^{-5}y^1(2z)^{-\frac{1}{2}}.$$

Problem 8. Solve the equation

$$x^2 - 4x - 21 = 0$$

by factoring the polynomial $x^2 - 4x - 21$.

Space for your solution:

The polynomial $x^2 - 4x - 21$ can be fit into the template of general trinomial $ax^2 + bx + c$ by taking $a = 1$, $b = -4$, $c = -21$.
Therefore

1. the discriminant $D = b^2 - 4ac = (-4)^2 - 4(1)(-21) = 16 + 84 = 100$;
2. the factorization

$$ax^2 + bx + c = a \cdot \left(x - \frac{-b + \sqrt{D}}{2a}\right)\left(x - \frac{-b - \sqrt{D}}{2a}\right)$$

becomes

$$x^2 - 4x - 21 = 1 \cdot \left(x - \frac{-(-4) + \sqrt{100}}{2(1)}\right)\left(x - \frac{-(-4) - \sqrt{100}}{2(1)}\right) = \left(x - \frac{4+10}{2}\right)\left(x - \frac{4-10}{2}\right) = (x-7)(x-(-3)) = (x-7)(x+3)$$

A product is zero if and only if one of the factors is zero. Therefore

$$x^2 - 4x - 21 = 0 \quad \Leftrightarrow \quad (x-7)(x+3) = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} x - 7 = 0 \\ x + 3 = 0 \end{array}\right. \quad \Leftrightarrow \quad \left[\begin{array}{l} x = 7 \\ x = -3 \end{array}\right.$$