Set Theory

Basic Concepts and Definitions

The Importance of Set Theory

One striking feature of humans is their inherent need – and ability – to group objects according to specific criteria. Our prehistoric ancestors grouped tools based on their hunting needs. They eventually evolved into strict hierarchical societies where a person belonged to one class and not another. Many of us today like to sort our clothes at house, or group the songs on our computer into playlists.

The idea of sorting out certain objects into similar groupings, or sets, is the most fundamental concept in modern mathematics. The theory of sets has, in fact, been the unifying framework for all mathematics since the German mathematician Georg Cantor formulated it in the 1870’s. No field of mathematics could be described nowadays without referring to some kind of abstract set. A geometer, for example, may study the set of parabolic curves in three dimensions or the set of spheres in a variety of different spaces. An algebraist may work with a set of equations or a set of matrices. A statistician typically works with large sets of raw data. And the list goes on. You may have also read or heard that the most important unresolved problem in mathematics at the moment deals with the set of prime numbers (this problem in number theory is known as Riemann’s Hypothesis; the Clay Institute will award a million dollars to whoever solves it.) As it turns out, even numbers are described by mathematicians in terms of sets!

More broadly, the concept of set membership, which lies at the heart of set theory, explains how statements with nouns and predicates are formulated in our language – or any abstract language like mathematics. Because of this, set theory is intimately connected to logic and serves as the foundation for all of mathematics.

What is a Set?

A set is a collection of objects called the elements or members of the set. These objects could be anything conceivable, including numbers, letters, colors, even sets themselves! However, none of the objects of the set can be the set itself. We discard this possibility to avoid running into Russell’s Paradox, a famous problem in mathematical logic unearthed by the great British logician Bertrand Russell in 1901.
Set Notation

We write sets using braces and denote them with capital letters. The most natural way to describe sets is by listing all its members.

For example, \( A = \{1,2,3,...,10\} \) is the set of the first 10 counting numbers, or naturals, \( B = \{\text{Red, Blue, Green}\} \) is the set of primary colors, \( N = \{1,2,3,...\} \) is the set of all naturals, and \( Z = \{..., -3, -2, -1, 0, 1, 2, 3,...\} \) is the set of all integers. Note the use of the ellipsis “…” to describe the infinite listings in the number sets \( N \) and \( Z \).

We write “\( x \in A \)” to mean “the object \( x \) belongs to the set \( A \)” and “\( x \notin A \)” to mean “the object \( x \) does not belong to the set \( A \)”.

Using the sets defined previously, we could then write “\( 1 \in A \)”, “\( 12 \notin A \)”, “\( 2012 \in N \)”, “\( 0 \notin N \)”, “\( \frac{2}{3} \notin Z \)” or “Black \( \notin B \)”.

Since many sets cannot be described by listing all its members (this can prove to be impossible), we also use the much more powerful set-builder, or predicate, notation. In that notation we write the set according to what types of objects belong to the set (these are placed to the left of the “\( | \)” which means “such that,” inside the braces) as well as the conditions that those objects must satisfy in order to belong to the set (these are placed to the right of the “\( | \)” inside the braces).

For example, the set of rational numbers, or fractions, which is denoted by \( Q \) cannot be described by a listing method. Instead, we write \( Q \) in set-builder notation as follows: \( Q = \left\{ \frac{p}{q} | p, q \in Z \text{ and } q \neq 0 \right\} \). This reads “\( Q \) is the set of all fractions of the form \( \frac{p}{q} \), such that \( p \) and \( q \) are integers and \( q \) is not zero.”

We could also write set \( A \) in our previous example as \( A = \{x | x \in N \text{ and } x < 11\} \) since all the elements of \( A \) (“\( x \)””) are natural numbers (“\( x \in N \)””) and none of these numbers are greater than 10 (“\( x < 11 \)””).

Well-Defined Sets

A set is said to be well-defined if it is unambiguous which elements belong to the set. In other words, if \( A \) is well-defined, then the question “Does \( x \in A \)” can always be answered for any object \( x \).

For example, all of our previous sets (\( A, B, N, Z, Q \)) are well-defined since it is perfectly clear which objects belong to each one of them. However if
we define C as the set of large numbers, then it is unclear which numbers should be considered “large”. C is therefore not a well-defined set. Similarly, the set of all great NY Mets players, or the set of all expensive restaurants in Riverhead, are also not well-defined.

**Number Sets**

Here are some important number sets used in mathematics:

- \( N = \{1,2,3,\ldots\} \) is the set of *counting numbers*, or *naturals*.
- \( W = \{0,1,2,3,\ldots\} \) is the set of *whole numbers*.
- \( Z = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\} \) is the set of *integers*.
- \( Q = \left\{ \frac{p}{q} \mid p,q \in Z \text{ and } q \neq 0 \right\} \) is the set of *rational numbers*, or *fractions*. This set can also be described as containing all terminating or non-terminating but repeating decimals.
- \( I \) is the set of *irrational numbers*. This set can also be described as containing all non-terminating and non-repeating decimals. Some of the most important numbers in mathematics belong to this set, including \( \pi, \sqrt{2}, e \) and \( \phi \).
- \( R \) is the set of *real numbers*. These are all the numbers that can be placed on a one-dimensional number line extending with no end on both the negative and positive sides.

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-5 -4 -3 -2 -1 0 1 2 3 4 5
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**Set Equality**

Two sets A and B are said to be *equal* (denoted, as expected, by \( A = B \)) if and only if both sets have the exact same elements. Here, *if and only if* means that both parts of the statement (“\( A = B \)” and “both sets have the exact same elements”) are interchangeable. Logically speaking, this means that each part of the statement implies the other.

Since we don’t care how the elements inside a set are listed, and since repetitions of elements inside a set are inconsequential, set equality can lead
to peculiar writings. For example, \( \{2,4,6,8\} = \{4,8,6,2\} \) and \( \{2,4,6,8\} = \{2,4,2,6,8,2,6,4,4\} \).

Another example comes from the set of even naturals, which can be described as \( E = \{2,4,6,8,\ldots\} = \{2x \mid x \in N\} \).

A very important set is the empty set, or the null set, which has no elements. We denote the empty set by \( \emptyset \), or \( \{\} \). Note that we could also write, for example, \( \emptyset = \{x \mid x \in N \text{ and } x < 0\} \) or \( \emptyset = \{x \mid x \in Q \text{ and } x \notin Q\} \).

**Cardinality of a Set**

The cardinality of a set \( A \) is the number of elements that belong to \( A \). We denote this number by \( n(A) \). The cardinality of a set can be informally thought of as a measure of its "size". If the cardinality of a set is a whole number, then the set is said to be finite. Otherwise, the set is said to be infinite. Georg Cantor, the German mathematician who founded set theory in the 1870's, came up with a revolutionary way to write the cardinality of infinite sets. This leads to what are called transfinite numbers and some higher-level concepts in set theory.

For example, the cardinality of the set of digits \( D = \{0,1,2,\ldots,8,9\} \) is 10. So we write \( n(D) = 10 \). The cardinality of the set \( \{2, 4, 6, \ldots, 18, 20\} \) is also 10.

The cardinality of the empty set is 0 since it contains no elements. So \( n(\emptyset) = 0 \). However, the cardinality of the set \( \{0\} \), which contains the number 0, is 1 since this set contains one element: the number 0.

**Subsets and Proper Subsets**

If all the elements of a set \( A \) are also elements of another set \( B \), then \( A \) is called a subset of \( B \). We denote this by \( A \subseteq B \). In a sense, you can think of subset \( A \) as being within, or contained in, set \( B \).

For example, the set of vowels \( \{a, e, i, o, u\} \) is a subset of the set of letters in the English alphabet. The set of women who are registered as independent voters is a subset of all eligible U.S. voters.

Since even naturals are special types of naturals (those divisible by 2), the set of even naturals \( E \) is a subset of \( N \). We then write \( E \subseteq N \).
You can check that the number sets enumerated above are all related as follows: \( \mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \). For example, \( \mathbb{Z} \subseteq \mathbb{Q} \) since every integer is also a fraction (with denominator 1).

If a set \( A \) is a subset of \( B \) and the two sets are not equal, then we call \( A \) a proper subset of \( B \). This is denoted by \( A \subset B \). Here, the set \( A \) is said to be properly contained in \( B \).

The set \( A = \{a, b, c, d, e\} \) is a subset of the set \( B = \{x \mid x \text{ is any of the first 5 letters in the alphabet}\} \), but \( A \) is not a proper subset of \( B \) since \( A = B \). We then write \( A \subseteq B \) but not \( A \subset B \).

Below are some important properties relating to subsets and proper subsets.

- Any set \( A \) is a subset of itself. Hence \( A \subseteq A \). This is clearly true.

- Less obvious is the fact that the empty set is a subset of any set \( A \) (i.e. \( \emptyset \subseteq A \)). You may conceive of this as stating that “nothing must belong to anything.” A better answer relies on the following proof by contradiction: start by assuming that this claim is false (so assume there exists a set \( A \) for which the empty set is not a subset). Then, according to the definition, the empty set must contain an element that is not in set \( A \)... but this is absurd since the empty set is empty! Hence, the initial assumption was wrong and so \( \emptyset \subseteq A \) for any set \( A \).

- The empty set is a proper subset of any set \( A \), provided that \( A \) is not the empty set itself.

- For finite sets \( A \) and \( B \), if \( A \subseteq B \) then \( n(A) \leq n(B) \).

- For finite sets \( A \) and \( B \), if \( A \subset B \) then \( n(A) < n(B) \).

**Power Sets**

The power set of a set \( A \), denoted by \( P_A \), is the set consisting of all distinct subsets of \( A \).

For example, the power set of the empty set is the set that contains the empty set: \( P_\emptyset = \{\emptyset\} \).

If \( A = \{I, II, III\} \), then \( P_A = \{\emptyset, \{I\}, \{II\}, \{III\}, \{I, II\}, \{I, III\}, \{II, III\}, A\} \) since \( A \) has exactly 8 subsets.
An important result in set theory states that if \( A \) is a set with \( k \) elements (i.e. \( n(A) = k \)), then the power set of \( A \) has exactly \( 2^k \) elements. We write this as follows: \( n(P_A) = 2^k \).

In our previous example, note that \( n(A) = 3 \) and so \( n(P_A) = 2^3 = 2 \cdot 2 \cdot 2 = 8 \).