Inductive Reasoning

The type of reasoning that forms a conclusion based on the examination of specific examples is called *inductive reasoning*. The conclusion formed by using inductive reasoning is often called a *conjecture*, since it may or may not be correct.

**Inductive Reasoning**

Inductive reasoning is the process of reaching a general conclusion by examining specific examples.

When you examine a list of numbers and predict the next number in the list according to some pattern you have observed, you are using inductive reasoning.

**Example 1** Use Inductive Reasoning to Predict a Number

Use inductive reasoning to predict the next number in each of the following lists.

a. 3, 6, 9, 12, 15, ?

b. 1, 3, 6, 10, 15, ?

**Solution**

a. Each successive number is 3 larger than the preceding number. Thus we predict that the next number in the list is 3 larger than 15, which is 18.

b. The first two numbers differ by 2. The second and the third numbers differ by 3. It appears that the difference between any two numbers is always 1 more than the preceding difference. Since 10 and 15 differ by 5, we predict that the next number in the list will be 6 larger than 15, which is 21.

**Check Your Progress** Use inductive reasoning to predict the next number in each of the following lists.

a. 5, 10, 15, 20, 25, ?

b. 2, 5, 10, 17, 26, ?

**Solution** See page 51.

Inductive reasoning is not used just to predict the next number in a list. In Example 2 we use inductive reasoning to make a conjecture about an arithmetic procedure.

**Example 2** Use Inductive Reasoning to Make a Conjecture

Consider the following procedure: Pick a number. Multiply the number by 8, add 6 to the product, divide the sum by 2, and subtract 3.

Complete the above procedure for several different numbers. Use inductive reasoning to make a conjecture about the relationship between the size of the resulting number and the size of the original number.

**Solution**

Suppose we pick 5 as our original number. Then the procedure would produce the following results:

- Original number: 5
- Multiply by 8: \(8 \times 5 = 40\)
**Take Note**

In Example 5, we will use a deductive method to verify that the procedure in Example 2 always yields a result that is four times the original number.

**Check Your Progress**

Consider the following procedure: Pick a number. Multiply the number by 9, add 15 to the product, divide the sum by 3, and subtract 5.

Complete the above procedure for several different numbers. Use inductive reasoning to make a conjecture about the relationship between the size of the resulting number and the size of the original number.

**Solution** See page S1.

Scientists often use inductive reasoning. For instance, Galileo Galilei (1564–1642) used inductive reasoning to discover that the time required for a pendulum to complete one swing, called the period of the pendulum, depends on the length of the pendulum. Galileo did not have a clock, so he measured the periods of pendulums in “heartbeats.” The following table shows some results obtained for pendulums of various lengths. For the sake of convenience, a length of 10 inches has been designated as 1 unit.

<table>
<thead>
<tr>
<th>Length of pendulum, in units</th>
<th>Period of pendulum, in heartbeats</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>36</td>
<td>6</td>
</tr>
</tbody>
</table>

The period of a pendulum is the time it takes for the pendulum to swing from left to right and back to its original position.

**Example** Use Inductive Reasoning to Solve an Application

Use the data in the table and inductive reasoning to answer each of the following questions.

a. If a pendulum has a length of 49 units, what is its period?

b. If the length of a pendulum is quadrupled, what happens to its period?

**Solution**

a. In the table, each pendulum has a period that is the square root of its length. Thus we conjecture that a pendulum with a length of 49 units will have a period of 7 heartbeats.

b. In the table, a pendulum with a length of 4 units has a period that is twice that of a pendulum with a length of 1 unit. A pendulum with a length of 16 units has a period that is twice that of a pendulum with a length of 4 units. It appears that quadrupling the length of a pendulum doubles its period.
A tsunami is a sea wave produced by an underwater earthquake. The height of a tsunami as it approaches land depends on the velocity of the tsunami. Use the table at the left and inductive reasoning to answer each of the following questions.

**Check Your Progress**

<table>
<thead>
<tr>
<th>Velocity of tsunami, in feet per second</th>
<th>Height of tsunami, in feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>18</td>
<td>36</td>
</tr>
<tr>
<td>21</td>
<td>49</td>
</tr>
<tr>
<td>24</td>
<td>64</td>
</tr>
</tbody>
</table>

**a.** What happens to the height of a tsunami when its velocity is doubled?

**b.** What should be the height of a tsunami if its velocity is 30 feet per second?

**Solution** See page 51.

Conclusions based on inductive reasoning may be incorrect. As an illustration, consider the circles shown below. For each circle, all possible line segments have been drawn to connect each dot on the circle with all the other dots on the circle.

The maximum numbers of regions formed by connecting dots on a circle

For each circle, count the number of regions formed by the line segments that connect the dots on the circle. Your results should agree with the results in the following table.

<table>
<thead>
<tr>
<th>Number of dots</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum number of regions</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>?</td>
</tr>
</tbody>
</table>

There appears to be a pattern. Each additional dot seems to double the number of regions. Guess the maximum number of regions you expect for a circle with six dots. Check your guess by counting the maximum number of regions formed by the line segments that connect six dots on a large circle. Your drawing will show that for six dots, the maximum number of regions is 31 (see the figure at the left), not 32 as you may have guessed. With seven dots the maximum number of regions is 57. This is a good example to keep in mind. Just because a pattern holds true for a few cases, it does not mean the pattern will continue. When you use inductive reasoning, you have no guarantee that your conclusion is correct.

**Counterexamples**

A statement is a true statement provided that it is true in all cases. If you can find one case for which a statement is not true, called a counterexample, then the statement is a false
statement. In Example 4 we verify that each statement is a false statement by finding a counterexample for each.

\[\text{\textbf{Example 4}}\quad \text{Find a Counterexample}\]

Verify that each of the following statements is a false statement by finding a counterexample.
For all numbers \(x\):
\[\begin{align*}
a. \quad |x| &> 0 \\
b. \quad x^2 &> x \\
c. \quad \sqrt{x^2} & = x
\end{align*}\]

\[\text{Solution}\]
A statement may have many counterexamples, but we need only find one counterexample to verify that the statement is false.
\[\begin{align*}
a. \quad \text{Let } x = 0. \text{ Then } |0| = 0. \text{ Because 0 is not greater than 0, we have found a counterexample. Thus } & \text{"for all numbers } x, |x| > 0\text{" is a false statement.} \\
b. \quad \text{For } x = 1 \text{ we have } 1^2 = 1. \text{ Since 1 is not greater than 1, we have found a counterexample. Thus } & \text{"for all numbers } x, x^2 > x\text{" is a false statement.} \\
c. \quad \text{Consider } x = -3. \text{ Then } \sqrt{(-3)^2} = \sqrt{9} = 3. \text{ Since 3 is not equal to } -3, \text{ we have found a counterexample. Thus } & \text{"for all numbers } x, \sqrt{x^2} = x\text{" is a false statement.}
\end{align*}\]

\[\text{\textbf{Check Your Progress 4}} \quad \text{Verify that each of the following statements is a false statement by finding a counterexample for each.} \]
For all numbers \(x\):
\[\begin{align*}
a. \quad \frac{x}{x} & = 1 \\
b. \quad \frac{x}{x+1} & = \frac{x}{x+1} \\
c. \quad \sqrt{x^2 + 16} & = x + 4
\end{align*}\]

\[\text{Solution} \quad \text{See page 81.}\]

\[\text{Question:} \quad \text{How many counterexamples are needed to prove that a statement is false?}\]

\[\text{Deductive Reasoning}\]
Another type of reasoning is called \textit{deductive reasoning}. Deductive reasoning is distinguished from inductive reasoning in that it is the process of reaching a conclusion by applying general principles and procedures.

\[\text{\textbf{Deductive Reasoning}}\]
\[\text{Deductive reasoning} \text{ is the process of reaching a conclusion by applying general assumptions, procedures, or principles.}\]

\[\text{\textbf{Example 5}} \quad \text{Use Deductive Reasoning to Establish a Conjecture}\]

Use deductive reasoning to show that the following procedure produces a number that is four times the original number.

\[\text{Answer: One}\]

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**Procedure:** Pick a number. Multiply the number by 8, add 6 to the product, divide the sum by 2, and subtract 3.

**Solution**

Let \( n \) represent the original number.

- Multiply the number by 8: \( 8n \)
- Add 6 to the product: \( 8n + 6 \)
- Divide the sum by 2: \( \frac{8n + 6}{2} = 4n + 3 \)
- Subtract 3: \( 4n + 3 - 3 = 4n \)

We started with \( n \) and ended with \( 4n \). The procedure given in this example produces a number that is four times the original number.

**Check your progress**

Use deductive reasoning to show that the following procedure produces a number that is three times the original number.

**Procedure:** Pick a number. Multiply the number by 6, add 10 to the product, divide the sum by 2, and subtract 5. **Hint:** Let \( n \) represent the original number.

**Solution** See page 51.

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**MATHMATTERS**

The MYST® Adventure Games and Inductive Reasoning

Most games have several written rules, and the players are required to use a combination of deductive and inductive reasoning to play the game. However, the MYST® computer/video adventure games have few written rules. Thus your only option is to explore and make use of inductive reasoning to discover the clues needed to solve the game.

---

**Inductive Reasoning vs. Deductive Reasoning**

In Example 6 we analyze arguments to determine whether they use inductive or deductive reasoning.

**Example 6** Determine Types of Reasoning

Determine whether each of the following arguments is an example of inductive reasoning or deductive reasoning.

a. During the past 10 years, a tree has produced plums every other year. Last year the tree did not produce plums, so this year the tree will produce plums.

b. All home improvements cost more than the estimate. The contractor estimated that my home improvement will cost $35,000. Thus my home improvement will cost more than $35,000.

**Solution**

a. This argument reaches a conclusion based on specific examples, so it is an example of inductive reasoning.
b. Because the conclusion is a specific case of a general assumption, this argument is an example of deductive reasoning.

**Check Your Progress** Determine whether each of the following arguments is an example of inductive reasoning or deductive reasoning.

a. All Janet Evanovich novels are worth reading. The novel *Twelve Sharp* is a Janet Evanovich novel. Thus *Twelve Sharp* is worth reading.

b. I know I will win a jackpot on this slot machine in the next 10 tries, because it has not paid out any money during the last 45 tries.

**Solution** See page 81.

**Logic Puzzles**

Logic puzzles, similar to the one in Example 7, can be solved by using deductive reasoning and a chart that enables us to display the given information in a visual manner.

**Example 7** Solve a Logic Puzzle

Each of four neighbors, Sean, Maria, Sarah, and Brian, has a different occupation (editor, banker, chef, or dentist). From the following clues, determine the occupation of each neighbor.

1. Maria gets home from work after the banker but before the dentist.
2. Sarah, who is the last to get home from work, is not the editor.
3. The dentist and Sarah leave for work at the same time.
4. The banker lives next door to Brian.

**Solution**

From clue 1, Maria is not the banker or the dentist. In the following chart, write X1 (which stands for “ruled out by clue 1”) in the Banker and the Dentist columns of Maria’s row.

<table>
<thead>
<tr>
<th>Editor</th>
<th>Banker</th>
<th>Chef</th>
<th>Dentist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sean</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maria</td>
<td></td>
<td>X1</td>
<td>X1</td>
</tr>
<tr>
<td>Sarah</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brian</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From clue 2, Sarah is not the editor. Write X2 (ruled out by clue 2) in the Editor column of Sarah’s row. We know from clue 1 that the banker is not the last to get home, and we know from clue 2 that Sarah is the last to get home; therefore, Sarah is not the banker. Write X2 in the Banker column of Sarah’s row.

<table>
<thead>
<tr>
<th>Editor</th>
<th>Banker</th>
<th>Chef</th>
<th>Dentist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sean</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maria</td>
<td></td>
<td>X1</td>
<td>X1</td>
</tr>
<tr>
<td>Sarah</td>
<td></td>
<td>X2</td>
<td>X2</td>
</tr>
<tr>
<td>Brian</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From clue 3, Sarah is not the dentist. Write X3 for this condition. There are now Xs for three of the four occupations in Sarah’s row; therefore, Sarah must be the chef. Place a √ in that box. Since Sarah is the chef, none of the other three people can be the chef. Write X3 for these conditions. There are now Xs for three of the four occupations in Maria’s row; therefore, Maria must be the editor. Insert a √ to indicate that Maria is the editor, and write X3 twice to indicate that neither Sean nor Brian is the editor.

<table>
<thead>
<tr>
<th>Editor</th>
<th>Banker</th>
<th>Chef</th>
<th>Dentist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sean</td>
<td></td>
<td></td>
<td>X3</td>
</tr>
<tr>
<td>Maria</td>
<td>√</td>
<td>X1</td>
<td>X3</td>
</tr>
<tr>
<td>Sarah</td>
<td>X2</td>
<td>X2</td>
<td></td>
</tr>
<tr>
<td>Brian</td>
<td>X3</td>
<td></td>
<td>X3</td>
</tr>
</tbody>
</table>

From clue 4, Brian is not the banker. Write X4 for this condition. Since there are three Xs in the Banker column, Sean must be the banker. Place a √ in that box. Thus Sean cannot be the dentist. Write X4 in that box. Since there are 3 Xs in the Dentist column, Brian must be the dentist. Place a √ in that box.

<table>
<thead>
<tr>
<th>Editor</th>
<th>Banker</th>
<th>Chef</th>
<th>Dentist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sean</td>
<td>X3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maria</td>
<td>X1</td>
<td>X3</td>
<td></td>
</tr>
<tr>
<td>Sarah</td>
<td>X2</td>
<td></td>
<td>X3</td>
</tr>
<tr>
<td>Brian</td>
<td>X3</td>
<td></td>
<td>√</td>
</tr>
</tbody>
</table>

Sean is the banker, Maria is the editor, Sarah is the chef, and Brian is the dentist.

**Check your progress** Brianna, Ryan, Tyler, and Ashley were recently elected as the new class officers (president, vice president, secretary, treasurer) of the sophomore class at Summit College. From the following clues, determine which position each holds.

1. Ashley is younger than the president but older than the treasurer.
2. Brianna and the secretary are both the same age, and they are the youngest members of the group.
3. Tyler and the secretary are next-door neighbors.

**Solution** See page 31.

**Excursion**

**KenKen® Puzzles: An Introduction**

KenKen® is an arithmetic-based logic puzzle that was invented by the Japanese mathematics teacher Tetsuya Miyamoto in 2004. The noun “ken” has “knowledge” and “awareness” as synonyms. Hence, KenKen translates as knowledge squared, or awareness squared.
In recent years the popularity of KenKen has increased at a dramatic rate. More than a million KenKen puzzle books have been sold, and KenKen puzzles now appear in many popular newspapers, including the *New York Times* and the *Boston Globe*.

KenKen puzzles are similar to Sudoku puzzles, but they also require you to perform arithmetic to solve the puzzle.

### Rules for Solving a KenKen Puzzle

For a 3 by 3 puzzle, fill in each box (square) of the grid with one of the numbers 1, 2, or 3.

For a 4 by 4 puzzle, fill in each square of the grid with one of the numbers 1, 2, 3, or 4.

For an *n* by *n* puzzle, fill in each square of the grid with one of the numbers 1, 2, 3, ..., *n*.

Grids range in size from a 3 by 3 up to a 9 by 9.

- Do not repeat a number in any row or column.
- The numbers in each heavily outlined set of squares, called *cages*, must combine (in some order) to produce the *target number* in the top left corner of the cage using the mathematical operation indicated.
- Cages with just one square should be filled in with the target number.
- A number can be repeated within a cage as long as it is not in the same row or column.

Here is a 4 by 4 puzzle and its solution. Properly constructed puzzles have a unique solution.

```
<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6×</td>
<td>2</td>
<td>8×</td>
<td>1</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>4×</td>
<td>12×</td>
<td>1−</td>
<td>1</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>
```

A 4 by 4 puzzle with 8 cages

```
<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>8×</td>
<td>4</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>4×</td>
<td>12×</td>
<td>1−</td>
<td>1</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

The solution to the puzzle

### Basic Puzzle Solution Strategies

**Single-Square Cages**  Fill cages that consist of a single square with the target number for that square.

**Cages with Two Squares**  Next examine the cages with exactly two squares. Many cages that cover two squares will only have two digits that can be used to fill the cage. For instance, in a 5 by 5 puzzle, a 20× cage with exactly two squares can only be filled with 4 and 5 or 5 and 4.

**Large or Small Target Numbers**  Search for cages that have an unusually large or small target number. These cages generally have only a few combinations of numbers that can be used to fill the cage.
Examples:
In a 6 by 6 puzzle, a 120× cage with exactly three squares can only be filled with 4, 5, and 6.

A 3+ cage with exactly two squares can only be filled with 1 and 2.

Duplicate Digit in a Cage Consider the 3× cage shown at the left. The digits 1, 1, and 3 produce a product of 3; however, we cannot place the two 1s in the same row or the same column. Thus the only way to fill the squares is to place the 3 in the corner of the L-shaped cage as shown below. Remember: A digit can occur more than once in a cage, provided that it does not appear in the same row or in the same column.

Remember the Following Rules
In an n by n puzzle, each row and column must contain every digit from 1 to n.
In a two-square cage that involves subtraction or division, the order of the numbers in the cage is not important. For instance, a 3− cage with two squares could be filled with 4 and 1 or with 1 and 4. A 3+ cage with two squares could be filled with 3 and 1 or with 1 and 3.

Make a List of Possible Digits For each cage, make a list of digits, with no regard to order, that can be used to fill the cage. See the following examples:

* In a 4 by 4 puzzle, this cage can only be filled with 1, 3, and 4 in some order. Note: 2, 2, and 3 cannot be used because the two 2s cannot be placed in the same row.

* In a 4 by 4 puzzle, this cage can be filled with 1, 1, 2, and 4, provided that the two 1s are placed in different rows. Note: The combination 1, 1, 3, and 3 and the combination 2, 2, 2, and 2 cannot be used because a duplicate digit would appear in the same row or column.

Guess and Check In most puzzles you will reach a point where you will need to experiment. Assume that the possible digits in a particular cage are arranged in a particular manner and then see where your assumption takes you. If you find that the remaining part of a row or column cannot be filled in correctly, then you can eliminate your assumption and proceed to check out one of the remaining possible numerical arrangements for that particular cage.

It is worth noting that there are generally several different orders that can be used to fill in the squares/cages in a KenKen puzzle, even though each puzzle has a unique solution.

Many Internet sites provide additional strategies for solving KenKen puzzles and you may benefit from watching some of the video tutorials that are available online. For instance, Will Shortz has produced a video tutorial on KenKen puzzles. It is available at http://www.wonderhowto.com/how-to-solve-kenken-puzzles-with-will-shortz

POINT OF INTEREST
Will Shortz, the well-known New York Times crossword puzzle editor, is reported to be the only person with a college degree in enigmatology, the study of puzzles.
EXCURSION EXERCISES

Solve each of the following puzzles. Note: The authors of this textbook are not associated with the KenKen brand. Thus, the following puzzles are not official KenKen puzzles; however, each puzzle can be solved using the same techniques one would use to solve an official KenKen puzzle.

1. \[ \begin{array}{c|c|c} 6+ & 3\times & 5+ \\ \hline & & \end{array} \]
2. \[ \begin{array}{c|c|c|c|c} 18\times & & 4+ & \hline & & \end{array} \]
3. \[ \begin{array}{c|c|c|c|c} 2\times & 64\times & 3 & \hline 8+ & 9+ & 3+ & 1 \end{array} \]
4. \[ \begin{array}{c|c|c|c|c} 3\times & 16\times & 3 & \hline 24\times & 8+ & 3- & 1 \end{array} \]
5. \[ \begin{array}{c|c|c|c|c|c} 160\times & 45\times & 5+ & \hline 20\times & 12+ & 4 & \end{array} \]
6. \[ \begin{array}{c|c|c|c|c|c} 40\times & 5+ & 4+ & \hline 36\times & 14+ & 4\times & \end{array} \]

EXERCISE SET 1.1

- In Exercises 1 to 10, use inductive reasoning to predict the next number in each list.

1. 4, 8, 12, 16, 20, 24, ?
2. 5, 11, 17, 23, 29, 35, ?
3. 3, 5, 9, 15, 23, 33, ?
4. 1, 8, 27, 64, 125, ?
5. 1, 4, 9, 16, 25, 36, 49, ?
6. 80, 70, 61, 53, 46, 40, ?
7. \[ \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \frac{9}{11}, \frac{11}{13}, \frac{13}{15}, ? \]
8. \[ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, ? \]
9. 2, 7, -3, 2, -8, -3, -13, -8, -18, ?
10. 1, 5, 12, 22, 35, ?
CHAPTER 1 | Problem Solving

There are many patterns that can be discovered in Pascal's triangle.

a. Find the sum of the numbers in each row, except row 0, of the portion of Pascal's triangle shown on page 27. What pattern do you observe concerning these sums? Predict the sum of the numbers in row 9 of Pascal's triangle.

b. The numbers 1, 3, 6, 10, 15, ..., $\frac{n(n + 1)}{2}$, ... are called triangular numbers. Where do the triangular numbers appear in Pascal's triangle?

36. A Savings Plan You save a penny on day 1. On each of the following days you save double the amount of money you saved on the previous day. How much money will you have after:

a. 5 days?  
b. 10 days?  
c. 15 days?  
d. $n$ days?  

*Hint: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, ..., $2^{10} = 1024$, ..., $2^{15} = 32768$, ...

Explorations

37. A Famous Puzzle The Tower of Hanoi is a puzzle invented by Edouard Lucas in 1883. The puzzle consists of three pegs and a number of disks of distinct diameters stacked on one of the pegs such that the largest disk is on the bottom, the next largest is placed on the largest disk, and so on as shown in the next figure.

The object of the puzzle is to transfer the tower to one of the other pegs. The rules require that only one disk be moved at a time and that a larger disk may not be placed on a smaller disk. All pegs may be used.

Determine the minimum number of moves required to transfer all of the disks to another peg for each of the following situations.

38. Use the recursive definition for Fibonacci numbers and deductive reasoning to verify that, for Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.  

*Hint: By definition, $F_{n+1} = F_n + F_{n-1}$ and $F_n = F_{n-1} + F_{n-2}$.

SECTION 13 | Problem-Solving Strategies

Polya's Problem-Solving Strategy

Ancient mathematicians such as Euclid and Pappus were interested in solving mathematical problems, but they were also interested in heuristics, the study of the methods and rules of discovery and invention. In the seventeenth century, the mathematician and philosopher René Descartes (1596–1650) contributed to the field of heuristics. He tried to develop a universal problem-solving method. Although he did not achieve this goal, he did publish some of his ideas in his book *Rules for the Direction of the Mind* and his better-known work *Discourse de la Methode*.

Another mathematician and philosopher, Gottfried Wilhelm Leibniz (1646–1716), planned to write a book on heuristics titled *Art of Invention*. Of the problem-solving process, Leibniz wrote, "Nothing is more important than to see the sources of invention which are, in my opinion, more interesting than the inventions themselves."

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One of the foremost recent mathematicians to make a study of problem solving was George Polya (1887–1985). He was born in Hungary and moved to the United States in 1940. The basic problem-solving strategy that Polya advocated consisted of the following four steps.

**Polya's Four-Step Problem-Solving Strategy**

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Review the solution.

Polya's four steps are deceptively simple. To become a good problem solver, it helps to examine each of these steps and determine what is involved.

**Understand the Problem** This part of Polya's four-step strategy is often overlooked. You must have a clear understanding of the problem. To help you focus on understanding the problem, consider the following questions.

- Can you restate the problem in your own words?
- Can you determine what is known about these types of problems?
- Is there missing information that, if known, would allow you to solve the problem?
- Is there extraneous information that is not needed to solve the problem?
- What is the goal?

**Devise a Plan** Successful problem solvers use a variety of techniques when they attempt to solve a problem. Here are some frequently used procedures.

- Make a list of the known information.
- Make a list of information that is needed.
- Draw a diagram.
- Make an organized list that shows all the possibilities.
- Make a table or a chart.
- Work backwards.
- Try to solve a similar but simpler problem.
- Look for a pattern.
- Write an equation. If necessary, define what each variable represents.
- Perform an experiment.
- Guess at a solution and then check your result.

**Carry Out the Plan** Once you have devised a plan, you must carry it out.

- Work carefully.
- Keep an accurate and neat record of all your attempts.
- Realize that some of your initial plans will not work and that you may have to devise another plan or modify your existing plan.

**Review the Solution** Once you have found a solution, check the solution.

- Ensure that the solution is consistent with the facts of the problem.
- Interpret the solution in the context of the problem.
- Ask yourself whether there are generalizations of the solution that could apply to other problems.
In Example 1 we apply Polya's four-step problem-solving strategy to solve a problem involving the number of routes between two points.

**Example 1: Apply Polya's Strategy** (Solve a similar but simpler problem)

Consider the map shown in Figure 1.2. Allison wishes to walk along the streets from point A to point B. How many direct routes can Allison take?

![City Map](image)

**Solution**

**Understand the Problem** We would not be able to answer the question if Allison retraced her path or traveled away from point B. Thus we assume that on a direct route, she always travels along a street in a direction that gets her closer to point B.

**Devise a Plan** The map in Figure 1.2 has many extraneous details. Thus we make a diagram that allows us to concentrate on the essential information. See the figure at the left.

Because there are many routes, we consider the similar but simpler diagrams shown below. The number at each street intersection represents the number of routes from point A to that particular intersection.

![Simple street diagrams](image)

The strategy of working a similar but simpler problem is an important problem-solving strategy that can be used to solve many problems.

Look for patterns. It appears that the number of routes to an intersection is the sum of the number of routes to the adjacent intersection to its left and the number of routes to the intersection directly above. For instance, the number of routes to the intersection labeled 6 is the sum of the number of routes to the intersection to its left, which is 3, and the number of routes to the intersection directly above, which is also 3.

**Carry Out the Plan** Using the pattern discovered above, we see from the figure at the left that the number of routes from point A to point B is 20 + 15 = 35.

**Review the Solution** Ask yourself whether a result of 35 seems reasonable. If you were required to draw each route, could you devise a scheme that would enable you to draw each route without missing a route or duplicating a route?
**Check your progress** Consider the street map in Figure 1.2. Allison wishes to walk directly from point A to point B. How many different routes can she take if she wants to go past Starbucks on Third Avenue?

**Solution** See page 52.

Example 2 illustrates the technique of using an organized list.

**Example 2.** Apply Polya’s Strategy (Make an organized list)

A baseball team won two out of their last four games. In how many different orders could they have two wins and two losses in four games?

**Solution**

**Understand the Problem** There are many different orders. The team may have won two straight games and lost the last two (WWLL). Or maybe they lost the first two games and won the last two (LLWW). Of course there are other possibilities, such as WLWL.

**Devise a Plan** We will make an organized list of all the possible orders. An organized list is a list that is produced using a system that ensures that each of the different orders will be listed once and only once.

**Carry Out the Plan** Each entry in our list must contain two Ws and two Ls. We will use a strategy that makes sure each order is considered, with no duplications. One such strategy is to always write a W unless doing so will produce too many Ws or a duplicate of one of the previous orders. If it is not possible to write a W, then only then do we write an L. This strategy produces the six different orders shown below.

1. WWLL (Start with two wins)
2. WLWL (Start with one win)
3. WLLW
4. LWWL (Start with one loss)
5. LWLW
6. LLWW (Start with two losses)

**Review the Solution** We have made an organized list. The list has no duplicates and the list considers all possibilities, so we are confident that there are six different orders in which a baseball team can win exactly two out of four games.

**Check your progress** A true-false quiz contains five questions. In how many ways can a student answer the questions if the student answers two of the questions with “false” and the other three with “true”?

**Solution** See page 53.

In Example 3 we make use of several problem-solving strategies to solve a problem involving the total number of games to be played.

**Example 3.** Apply Polya’s Strategy (Solve a similar but simpler problem)

In a basketball league consisting of 10 teams, each team plays each of the other teams exactly three times. How many league games will be played?
Solution

Understand the Problem  There are 10 teams in the league, and each team plays exactly three games against each of the other teams. The problem is to determine the total number of league games that will be played.

Devise a Plan  Try the strategy of working a similar but simpler problem. Consider a league with only four teams (denoted by A, B, C, and D) in which each team plays each of the other teams only once. The diagram at the left illustrates that the games can be represented by line segments that connect the points A, B, C, and D.

Since each of the four teams will play a game against each of the other three, we might conclude that this would result in \(4 \cdot 3 = 12\) games. However, the diagram shows only six line segments. It appears that our procedure has counted each game twice. For instance, when team A plays team B, team B also plays team A. To produce the correct result, we must divide our previous result, 12, by 2. Hence, four teams can play each other once in \(\frac{4 \cdot 3}{2} = 6\) games.

Carry Out the Plan  Using the process developed above, we see that 10 teams can play each other once in a total of \(\frac{10 \cdot 9}{2} = 45\) games. Since each team plays each opponent exactly three times, the total number of games is \(45 \cdot 3 = 135\).

Review the Solution  We could check our work by making a diagram that includes all 10 teams represented by dots labeled A, B, C, D, E, F, G, H, I, and J. Because this diagram would be somewhat complicated, let’s try the method of making an organized list. The figure at the left shows an organized list in which the notation BC represents a game between team B and team C. The notation CB is not shown because it also represents a game between team B and team C. This list shows that 45 games are required for each team to play each of the other teams once. Also notice that the first row has nine items, the second row has eight items, the third row has seven items, and so on. Thus 10 teams require

\[9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45\]

games if each team plays every other team once, and \(45 \cdot 3 = 135\) games if each team plays exactly three games against each opponent.


In Example 4 we make use of a table to solve a problem.

\[\text{Example 4} \quad \text{Apply Polya's Strategy} \quad \text{(Make a table and look for a pattern)}\]

Determine the digit 100 places to the right of the decimal point in the decimal representation \(\frac{7}{27}\).

Solution

Understand the Problem  Express the fraction \(\frac{7}{27}\) as a decimal and look for a pattern that will enable us to determine the digit 100 places to the right of the decimal point.

Devise a Plan  Dividing 27 into 7 by long division or by using a calculator produces the decimal 0.259259259259259259... Since the decimal representation repeats the digits 259 over and over forever, we know that the digit located 100 places to the right of the decimal point is 2.
point is either 2, a 5, or a 9. A table may help us to see a pattern and enable us to
determine which one of these digits is in the 100th place. Since the decimal digits repeat
every three digits, we use a table with three columns.

<table>
<thead>
<tr>
<th>Column 1 Location</th>
<th>Digit</th>
<th>Column 2 Location</th>
<th>Digit</th>
<th>Column 3 Location</th>
<th>Digit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2</td>
<td>2nd</td>
<td>5</td>
<td>3rd</td>
<td>9</td>
</tr>
<tr>
<td>4th</td>
<td>2</td>
<td>5th</td>
<td>5</td>
<td>6th</td>
<td>9</td>
</tr>
<tr>
<td>7th</td>
<td>2</td>
<td>8th</td>
<td>5</td>
<td>9th</td>
<td>9</td>
</tr>
<tr>
<td>10th</td>
<td>2</td>
<td>11th</td>
<td>5</td>
<td>12th</td>
<td>9</td>
</tr>
<tr>
<td>13th</td>
<td>2</td>
<td>14th</td>
<td>5</td>
<td>15th</td>
<td>9</td>
</tr>
</tbody>
</table>

**Carry Out the Plan**  Only in column 3 is each of the decimal digit locations evenly
divisible by 3. From this pattern we can tell that the 99th decimal digit (because 99 is
evenly divisible by 3) must be a 9. Since a 2 always follows a 9 in the pattern, the 100th
decimal digit must be a 2.

**Review the Solution**  The above table illustrates additional patterns. For instance, if
each of the location numbers in column 1 is divided by 3, a remainder of 1 is produced.
If each of the location numbers in column 2 is divided by 3, a remainder of 2 is produced.
Thus we can find the decimal digit in any location by dividing the location number by 3
and examining the remainder. For instance, to find the digit in the 3200th decimal place
of \( \frac{7}{27} \), merely divide 3200 by 3 and examine the remainder, which is 2. Thus, the digit
3200 places to the right of the decimal point is a 5.

**Check Your Progress**  Determine the units digit (ones digit) of \( 4^{200} \).

**Solution**  See page 83.

Example 5 illustrates the method of working backwards. In problems in which you
know a final result, this method may require the least effort.

**Example 6**  Apply Polya's Strategy  (Work backwards)

In consecutive turns of a Monopoly game, Stacy first paid $800 for a hotel. She then lost
half her money when she landed on Boardwalk. Next, she collected $200 for passing
GO. She then lost half her remaining money when she landed on Illinois Avenue. Stacy
now has $2500. How much did she have just before she purchased the hotel?

**Solution**

*Understand the Problem*  We need to determine the number of dollars that Stacy had
just prior to her $800 hotel purchase.

*Devise a Plan*  We could guess and check, but we might need to make several guesses
before we found the correct solution. An algebraic method might work, but setting up
the necessary equation could be a challenge. Since we know the end result, let's try the
method of working backwards.
**Carry Out the Plan**  Stacy must have had $5000 just before she landed on Illinois Avenue; $4800 just before she passed GO; and $9600 prior to landing on Boardwalk. This means she had $10,400 just before she purchased the hotel.

**Review the Solution**  To check our solution we start with $10,400 and proceed through each of the transactions: $10,400 less $800 is $9600. Half of $9600 is $4800. $4800 increased by $200 is $5000. Half of $5000 is $2500.

**Check Your Progress**  Melody picks a number. She doubles the number, squares the result, divides the square by 3, subtracts 30 from the quotient, and gets 18. What are the possible numbers that Melody could have picked? What operation does Melody perform that prevents us from knowing with 100% certainty which number she picked?

**Solution**  See page S3.

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**MATHMATTERS**  A Mathematical Prodigy

Karl Friedrich Gauss (1777-1855) was a scientist and mathematician. His work encompassed several disciplines, including number theory, analysis, astronomy, and optics. He is known for having shown mathematical prowess as early as age three. It is reported that soon after Gauss entered elementary school, his teacher assigned the problem of finding the sum of the first 100 natural numbers. Gauss was able to determine the sum in a matter of a few seconds. The following solution shows the thought process he used.

**Understand the Problem**  The sum of the first 100 natural numbers is represented by

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

**Devise a Plan**  Adding the first 100 natural numbers from left to right would be time consuming. Gauss considered another method. He added 1 and 100 to produce 101. He noticed that 2 and 99 have a sum of 101, and that 3 and 98 have a sum of 101. Thus the 100 numbers could be thought of as 50 pairs, each with a sum of 101.

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

$$\begin{array}{c}
101 \\
\hline
101 \\
\hline
101
\end{array}$$

**Carry Out the Plan**  To find the sum of the 50 pairs, each with a sum of 101. Gauss computed $50 \cdot 101$ and arrived at 5050 as the solution.

**Review the Solution**  Because the addends in an addition problem can be placed in any order without changing the sum, Gauss was confident that he had the correct solution.

**An Extension**  The sum $1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$ can be found by using the following formula.

\[1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = \frac{n(n+1)}{2}\]
Some problems can be solved by making guesses and checking. Your first few guesses may not produce a solution, but quite often they will provide additional information that will lead to a solution.

**Example 5** Apply Polya's Strategy (Guess and check)

The product of the ages, in years, of three teenagers is 4590. None of the teens are the same age. What are the ages of the teenagers?

**Solution**

*Understand the Problem* We need to determine three distinct counting numbers, from the list 13, 14, 15, 16, 17, 18, and 19, that have a product of 4590.

*Devise a Plan* If we represent the ages by \( x, y, \) and \( z \), then \( xyz = 4590 \). We are unable to solve this equation, but we notice that 4590 ends in a zero. Hence, 4590 has a factor of 2 and a factor of 5, which means that at least one of the numbers we seek must be an even number and at least one number must have 5 as a factor. The only number in our list that has 5 as a factor is 15. Thus 15 is one of the numbers, and at least one of the other numbers must be an even number. At this point we try to solve by guessing and checking.

*Carry Out the Plan*

\[
15 \cdot 16 \cdot 18 = 4320 \quad \text{• No. This product is too small.}
\]

\[
15 \cdot 16 \cdot 19 = 4560 \quad \text{• No. This product is too small.}
\]

\[
15 \cdot 17 \cdot 18 = 4590 \quad \text{• Yes. This is the correct product.}
\]

The ages of the teenagers are 15, 17, and 18.

*Review the Solution* Because \( 15 \cdot 17 \cdot 18 = 4590 \) and each of the ages represents the age of a teenager, we know our solution is correct. None of the numbers 13, 14, 16, and 19 is a factor (divisor) of 4590, so there are no other solutions.

Nothing is known about the personal life of the ancient Greek mathematician Diophantus except for the information in the following epigram. “Diophantus passed \( \frac{1}{6} \) of his life in childhood, \( \frac{1}{12} \) in youth, and \( \frac{1}{7} \) more as a bachelor. Five years after his marriage was born a son who died four years before his father, at \( \frac{1}{5} \) his father’s (final) age.”

A diagram of the data, where \( x \) represents the age of Diophantus when he died

How old was Diophantus when he died? (Hint: Although an equation can be used to solve this problem, the method of guessing and checking will probably require less effort. Also assume that his age, when he died, is a counting number.)

**Solution** See page S3.
Is the process of guessing at a solution and checking your result one of Polya's problem-solving strategies?

Some problems are deceptive. After reading one of these problems, you may think that the solution is obvious or impossible. These deceptive problems generally require that you carefully read the problem several times and that you check your solution to make sure it satisfies all the conditions of the problem.

**Example 7** Solve a Deceptive Problem

A hat and a jacket together cost $100. The jacket costs $90 more than the hat. What are the cost of the hat and the cost of the jacket?

**Solution**

*Understand the Problem* After reading the problem for the first time, you may think that the jacket costs $90 and the hat costs $10. The sum of these costs is $100, but the cost of the jacket is only $80 more than the cost of the hat. We need to find two dollar amounts that differ by $90 and whose sum is $100.

*Devise a Plan* Write an equation using $h$ for the cost of the hat and $h + 90$ for the cost of the jacket.

\[ h + h + 90 = 100 \]

*Carry Out the Plan* Solve the above equation for $h$.

\[ 2h + 90 = 100 \]
\[ 2h = 10 \]
\[ h = 5 \]

The cost of the hat is $5 and the cost of the jacket is $90 + $5 = $95.

*Review the Solution* The sum of the costs is $5 + $95 = $100, and the cost of the jacket is $90 more than the cost of the hat. This check confirms that the hat costs $5 and the jacket costs $95.

**Check Your Progress 7** Two U.S. coins have a total value of 35¢. One of the coins is not a quarter. What are the two coins?

**Solution** See page 3.

**Reading and Interpreting Graphs**

Graphs are often used to display numerical information in a visual format that allows the reader to see pertinent relationships and trends quickly. Three of the most common types of graphs are the bar graph, the circle graph, and the broken-line graph.

Figure 1.3 is a bar graph that displays the average U.S. movie theatre ticket prices for the years from 2003 to 2009. The years are displayed on the horizontal axis. Each vertical bar is used to display the average ticket price for a given year. The higher the bar, the greater the average ticket price for that year.

**Answer** Yes.
**Figure 1.4** is a circle graph or pie chart that uses circular sectors to display the percent of the films, released in 2009, that received a particular rating. The size of a sector is proportional to the percent of films that received the rating shown by its label.

Figure 1.5 shows two broken-line graphs. The red broken-line graph displays the median age at first marriage for men for the years from 2001 to 2009. The green broken-line graph displays the median age at first marriage for women during the same time period. The symbol on the vertical axis indicates that the ages between 0 and 25 are not displayed. This break in the vertical axis allows the graph to be displayed in a compact form. The segments that connect points on the graph indicate trends. Increasing trends are indicated by segments that rise as they move to the right, and decreasing trends are indicated by segments that fall as they move to the right. The blue arrows in Figure 1.5 show that the median age at which men married for the first time in 2006 was 27.5 years, rounded to the nearest half of a year.

---

**Example 1** Use Graphs to Solve Problems

a. Use Figure 1.3 to determine the minimum average U.S. movie theatre ticket price for the years from 2003 to 2009.

b. Use Figure 1.4 to determine the number of films released in 2009 that received a PG-13 rating. Round to the nearest counting number. *Note:* 511 movies were released in 2009.

c. Use Figure 1.5 to estimate the median age at which women married for the first time in 2008. Round to the nearest half of a year.

**Solution**

a. The minimum of the average ticket prices is displayed by the height of the shortest vertical bar in Figure 1.3. Thus the minimum average U.S. movie theatre ticket price for the years from 2003 to 2009 was $6.03, in the year 2003.

b. Figure 1.4 indicates that 27.0% of the 511 films released in 2009 received a PG-13 rating. Thus $0.27 \times 511 \approx 138$ of the films received a PG-13 rating in 2009.

c. To estimate the median age at which women married for the first time in 2008, locate 2008 on the horizontal axis of Figure 1.5 and then move directly upward to a point on the green broken-line graph. The height of this point represents the median age at first marriage for women in 2008, and it can be estimated by moving horizontally to the vertical axis on the left. Thus the median age at first marriage for women in 2008 was 26.0 years, rounded to the nearest half of a year.