Continuous Income Streams

JANE-MARIE WRIGHT

Fall 2009

Abstract. MAT 131 lecture notes for Section 7.4

1. Continuous Income Streams

1.1. Total Value of a Continuous Income Stream. NOTE: All page/problem numbers need to be updated

If the rate of receipt of income is $R(t)$ dollars per unit of time, then the total income received from time $t = a$ to $t = b$ is

Total value $= TV = \int_a^b R(t)dt$

Example 1. Page 405 Problem 2. Find the total value of the income stream $R(t) = 40000$ from time $t = 0$ to $t = 5$

\[\int_0^5 40000\,dt = 40000(5) = 200000\]

Example 2. Page 405 Problem 6. Find the total value of the income stream $R(t) = 40000e^{0.04t}$ from time $t = 0$ to $t = 5$

\[\int_0^5 40000e^{0.04t}\,dt = 1000000e^{0.04t} - 1000000 = 221402.76\]

Example 3. Page 405 Problem 18 The rate of receipt of after-tax profit by South African Breweries form 1991 to 1997 can be modeled by the function $P(t) = 7.0t^2 - 9.5t + 350$ million dollars per years ($1 \leq t \leq 7$) where $t$ is the time in years since January 1990. Use this as a model of a continuous income stream to estimate SAB’s total after tax profits from January 1991 through January 1997.

\[TP = \int_a^b P(t)\,dt = \int_1^7 (7.0t^2 - 9.5t + 350)\,dt\]

\[= \left[2.3333t^3 - 4.75t^2 + 350.0t\right]_1^7\]

\[= \left[2.3333 \cdot 7^3 - 4.75 \cdot 7^2 + 350.0 \cdot 7\right] - \left[2.3333 - 4.75 + 350.0\right]\]

\[= 2670\]
1.2. Future Value of a Continuous Income Stream. If the rate of receipt of income from time \( t = a \) to \( t = b \) is \( R(t) \) dollars per unit of time, and the income is deposited as it is received in an account paying interest \( r \) per unit time, compounded continuously, then the amount of money in the account at time \( t = b \) is

Future value \( = FV = \int_a^b R(t)e^{r(b-t)} \, dt = e^{rb} \int_a^b R(t)e^{-rt} \, dt \)

Example 4. Page 405 Problem 2. Find the future value of the income stream \( R(t) = 40000 \) from time \( t = 0 \) to \( t = 5 \) when \( r = 10\% \)

\[
e^{rb} \int_a^b R(t)e^{-rt} \, dt = e^{5(1)} \int_0^5 40000e^{-0.1t} \, dt = -400000e^{-0.1t}\bigg|_0^5 = -400000e^{-0.5} (e^{-0.5} - 1) = 259488.50
\]

Example 5. Page 405 Problem 2. Find the future value of the income stream \( 40000e^{0.04t} \) from time \( t = 0 \) to \( t = 5 \) when \( r = 10\% \)

\[
e^{rb} \int_a^b R(t)e^{-rt} \, dt = e^{5(1)} \int_0^5 40000e^{0.04t}e^{-0.1t} \, dt = 40000e^5 \int_0^5 e^{-0.06t} \, dt = \frac{-40000}{0.06}e^{-0.06t}\bigg|_0^5 = \frac{-40000}{0.06}e^5 (e^{-0.3} - 1) = 284879.01
\]

1.3. Present Value of a Continuous Income Stream. If the rate of receipt of income from time \( t = a \) to \( t = b \) is \( R(t) \) dollars per unit of time, and the income is deposited as it is received in an account paying interest \( r \) per unit time, compounded continuously, then the value of the income stream at time \( t = a \) is

Present value \( = PV = \int_a^b R(t)e^{r(a-t)} \, dt = e^{ra} \int_a^b R(t)e^{-rt} \, dt \)

Remark 1. \( FV = PV e^{r(b-a)} \)

Example 6. Page 405 Problem 12. Find the total value of the income stream \( R(t) = 50000 + 2000t \) and find its present value at the end of the interval \( 0 \leq t \leq 10 \) using interest rate \( 5\% \)

Total value \( = TV = \int_0^{10} (50000 + 2000t) \, dt = [50000t + 1000t^2]_0^{10} = 50000 \cdot 10 + 1000 \cdot 10^2 = 600000 \)

Present value \( = PV = FV e^{r(10)} \)
The first part is easy, the second part requires integration by parts

\[ PV = e^{ra} \int_a^b R(t)e^{-rt} \, dt = e^{0} \int_0^{10} (50000 + 2000t) e^{-0.05t} \, dt \]

\[ = 50000 \int_0^{10} e^{-0.05t} \, dt + 2000 \int_0^{10} te^{-0.05t} \, dt \]

\[ 50000 \int_0^{10} e^{-0.05t} \, dt = 50000 \frac{e^{-0.05t}}{-0.05} \bigg|_0^{10} \]

\[ = -1000000e^{-0.5} + 1000000 = 393469.34 \]

\[ 2000 \int_0^{10} te^{-0.05t} \, dt \]

\[ u = t \text{ : } dv = e^{-0.05t} \, dt \]

\[ du = dt \text{ : } v = -20e^{-0.05t} \]

\[ -20te^{-0.05t} - \int -20e^{-0.05t} \, dt \]

\[ -20te^{-0.05t} - 400e^{-0.05t} \]

\[ 2000 \int_0^{10} te^{-0.05t} \, dt = 2000 \left( -20te^{-0.05t} - 400e^{-0.05t} \right)_{10} \]

\[ = 2000 \left( -200e^{-0.5} - 400e^{-0.5} \right) - 2000 (-400) \]

\[ = 72163.21 \]

\[ PV = 50000 \int_0^{10} e^{-0.05t} \, dt + 2000 \int_0^{10} te^{-0.05t} \, dt \]

\[ = 393469.34 + 72163.21 = 465632.55 \]


Page 406 Problem 26. When your first child is born, you begin to save for college by depositing $400 per month in an account paying 12% interest per year. With a continuous steam of investment and continuous compounding, how much will you have accumulated in the account by the time your child enters college 18 years later?

Notice that you are putting money away per month. 18 \cdot 12 = 216 months. The interest rate should them be 12% \div 12 = 1% per month

\[ R(t) = 400 \]

\[ FV = e^{rb} \int_a^b R(t)e^{-rt} \, dt = 400e^{2.16} \int_0^{216} e^{-0.01t} \, dt \]

\[ = -40000e^{2.16} \left( e^{-2.16} - 1 \right) = 306845.51 \]
Page 406 Problem 28. When your first child is born, you begin to save for college by depositing $400 per month in an account paying 12% interest per year. You increase the amount you save by 2% per year. With a continuous stream of investment and continuous compounding, how much will you have accumulated in the account by the time your child enters college 18 years later?

\[ R(t) = 400e^{\frac{0.02}{12}t} \]

\[ FV = 400e^{2.16} \int_{0}^{216} e^{\frac{1}{1200}t} e^{-0.01t} \, dt = 400e^{2.16} \int_{0}^{216} e^{\frac{-t}{120}} \, dt \]

\[ = \left[ -48000e^{2.16}e^{\frac{-t}{120}} \right]_{0}^{216} = -48000e^{2.16}e^{\frac{-216}{120}} + 48000e^{2.16} \]

\[ = 347414.80 \]